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Diffraction By An Elliptic Cone

LESTER KRAUS and LEO M. LEVINE

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Abstract

The boundary value problems corresponding to a spherical wave incident on an acoustically soft or hard elliptic cone, or plane angular sector, are solved. The method of separation of variables in a sphero-conal coordinate system is used. The special functions which arise are simply periodic and nonperiodic solutions of the Lamé differential equation.

Table of Contents

	Page
1. Introduction	1
2. The reduced wave equation in sphero-conal coordinates	2
3. Green's functions	9
Appendix I: The sphero-conal coordinate system	16
Appendix II: Orthogonality and completeness of the Lamé products	19
References	25

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1. Introduction.

In this article we shall derive two Green's functions for the reduced wave equation in the exterior (or interior) of an elliptic cone. One of them and the normal derivative of the other vanish on the surface of the elliptic cone. Both satisfy the radiation condition. Thus they represent the fields produced by a point source and diffracted by a soft or hard cone respectively. Since a plane angular sector is a degenerate elliptic cone, the problem of diffraction by a plane angular sector is included as a most important special case. These solutions can be used to determine the behavior of the field near the vertex, this being carried out in [1]. They can also be used to obtain the diffraction coefficient of the vertex which is needed in Keller's geometrical theory of diffraction [2,3].

The Green's functions are constructed by the method of separation of variables in a suitable coordinate system. This system, introduced in [4], is a 'uniformized' version of the sphero-conal coordinate system. In the analysis, there are several features not usually encountered in separable problems. First, the singular surfaces of the coordinate system play an important role in determining the boundary conditions for the resulting Sturm-Liouville problems. Second, the two separation constants occur in both angular Sturm-Liouville equations. Therefore they must be determined simultaneously. In this respect the problem resembles that of determining the vibrations of an elliptical membrane.

The special functions which arise are simply periodic and nonperiodic solutions of the Lamé differential equation. The doubly periodic Lamé functions, which occur in classical potential problems for ellipsoids, have been the subject of numerous investigations. However, the simply periodic Lamé functions have not been studied extensively while the nonperiodic solutions, which we shall call nonperiodic Lamé functions, have apparently not been studied at all.

We will formulate our diffraction problems as follows. Let the domain R be the exterior or interior of a semi-infinite elliptic cone, or the exterior of a plane angular sector, and let C be the boundary of R. We wish to find the two Green's functions $G^{(1)}(\vec{r},\vec{r}\,')$ and $G^{(2)}(\vec{r},\vec{r}\,')$ defined for $\vec{r},\vec{r}\,'\epsilon R+C$, $\vec{r} \neq \vec{r}\,'$ which for fixed $\vec{r}\,'$ satisfy the following conditions as functions of $\vec{r}$:

(a) $\Delta G^{(1,2)} + \kappa^2 G^{(1,2)} = -\delta(\vec{r}-\vec{r}\,')$

(b) $G^{(1,2)} \epsilon C^{(2)}$ in $R-\vec{r}\,'$, $G^{(1,2)} \epsilon C^{(o)}$ in $R + C-\vec{r}\,'$

(c) $G^{(1)} = 0$ on C

$\frac{\partial G^{(2)}}{\partial n} = 0$ and grad $G^{(2)}$ exists at regular points of C

(d) $G^{(1,2)}$ satisfies the radiation condition

$$\lim_{\rho \to \infty} \int_{\Omega_\rho} \left| \frac{\partial G^{(1,2)}}{\partial r} - i\kappa G^{(1,2)} \right|^2 d\Omega = 0$$

Here Ω_ρ is the part contained in R of the surface of the sphere of radius ρ centered at the vertex of C and $r = |\vec{r}|$. The fact that these conditions determine $G^{(1,2)}$ uniquely has been proved in [5].

In section 2, we describe the sphero-conal system, discuss its geometry, and reformulate our diffraction problem in terms of these coordinates. In section 3 we apply the method of separation of variables, discuss the resulting ordinary differential equations and derive the Green's functions. The construction of the sphero-conal coordinate system is described in detail in Appendix I, and a proof of the completeness and orthogonality of the Lame products, which comprise the angular eigenfunctions,is given in Appendix II. The proof is carried out along lines indicated in [4].

2. The reduced wave equation in sphero-conal coordinates.

Let C be a semi-infinite elliptic cone with vertex at the origin and a directrix consisting of an ellipse in a plane x = const. with its center on

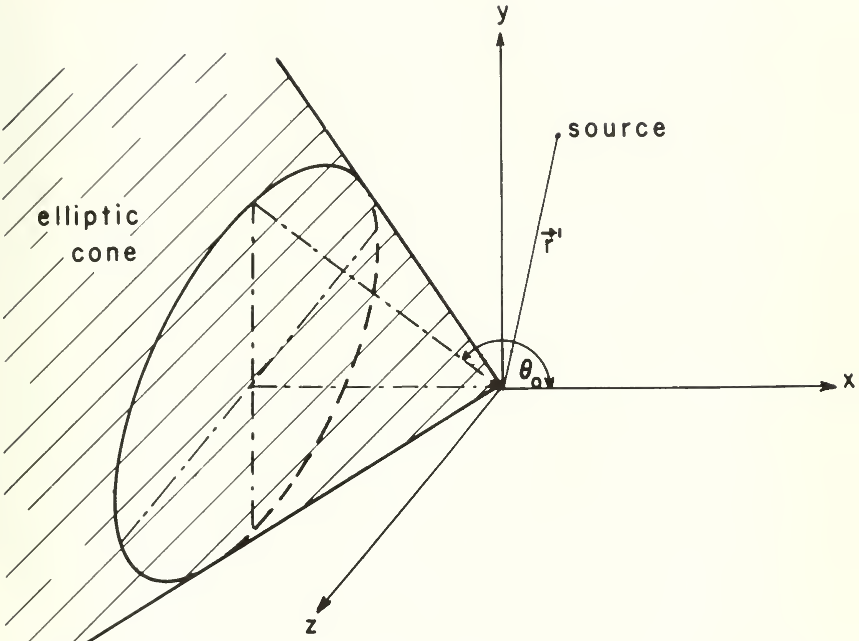

Figure 1

The domain of the diffraction problem is the exterior of a semi-infinite elliptic cone. The elliptic directrix is shown. ϑ_o is the angle between the generatrix through the end of the minor axis and the positive x-axis.

the x-axis and its major axis in the plane $y = 0$. (See Figure 1.) Let R be that domain bounded by C which contains the positive x-axis, and let ϑ_0 be the angle between the positive x-axis and the generatrix through the end of the minor axis of the elliptic directrix. Thus R is the exterior or interior of an elliptic cone, depending on whether $\frac{\pi}{2} < \vartheta_0 < \pi$, or $0 < \vartheta_0 < \frac{\pi}{2}$ Let ϵ be the semi-angle subtended at the origin by the foci of the ellipse and let

$$k = (1 + \tan^2\epsilon \cos^2\vartheta_0)^{-1/2}$$

(Here, and in what follows, square roots are always to be taken as positive.)

We introduce sphero-conal coordinates r, ϑ, φ by the equations

$$\begin{aligned} x &= r \cos \vartheta(1 - k'^2 \cos^2\varphi)^{1/2} \\ y &= r \sin \vartheta \sin \varphi \\ z &= r \cos \varphi(1 - k^2\cos^2\vartheta)^{1/2} \end{aligned} \tag{2.1}$$

Here, $k' = (1 - k^2)^{1/2}$, and the ranges of the variables are given by

$$0 \leqq \vartheta \leqq \pi$$

$$0 \leqq \varphi \leqq 2\pi$$

$$r \geqq 0$$

The construction of this coordinate system is described, and its orthogonality proved in Appendix I. The geometry of the coordinate surfaces may be described briefly as follows [see Figure 2]. The coordinate r is the distance to the origin, so that the set of coordinate surfaces $r = \text{const.}$ is, of course, the set of spheres centered at the origin.

The coordinate surface $\vartheta = \text{const.}$ is a semi-infinite elliptic cone whose cross section in a plane $x = \text{const.}$ is an ellipse centered on the x-axis, with its major axis in the plane $y = 0$. ϑ has a simple geometrical interpretation as the angle between the generatrix through the end of the minor axis of any cross sectional ellipse and the positive x-axis. The boundary

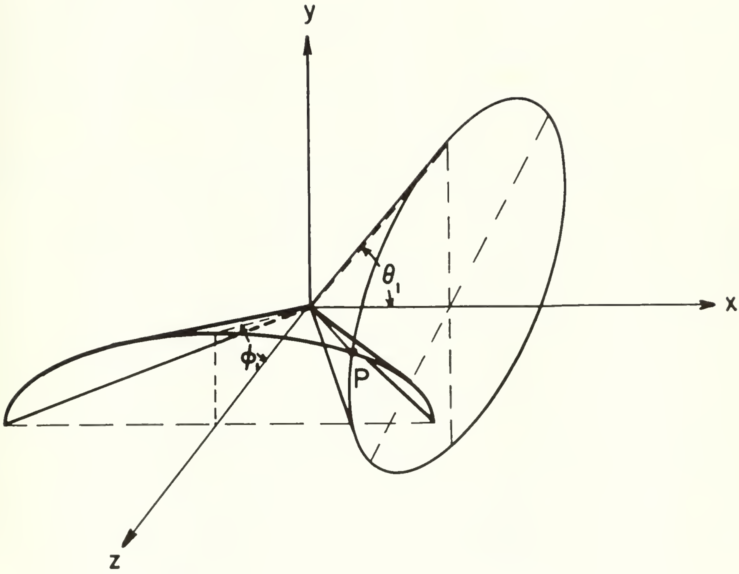

Figure 2

The Sphero-conal Coordinate System. The sphero-conal angular coordinates ϑ_1, φ_1 of the point P are shown, together with the coordinate surface $\vartheta = \vartheta_1$, an elliptic cone, and the surface $\varphi = \varphi_1$ which is half of an elliptic cone. The cones are shown truncated by (cartesian) coordinate planes through P. The sphere though P, with center at the origin, is not shown.

cone C is given by the equation $\vartheta = \vartheta_o$.

The surface φ = const. is a semi-infinite elliptic half-cone whose cross section in a plane z = const. is half of an ellipse centered on the z-axis, with its major axis in the plane y = 0. The semi-ellipse lies above the plane y = 0 if $0 < \varphi < \pi$ and below the plane if $\pi < \varphi < 2\pi$. The coordinate φ has a geometrical interpretation like that of ϑ, except that φ takes on values greater than π when the generatrix through the end of the semi-minor axis (and the semi-ellipse) lies below the plane y = 0.

It is evident that the coordinate system defined by (2.1) becomes the familiar spherical coordinate system when k = 1. Thus, when the semi-focal angle ϵ of C vanishes, C is a circular cone. Then ϑ becomes the polar angle, as usual, and the semi-cones φ = const. become half-planes.

For $k \neq 1$, the coordinate surfaces $\vartheta = 0$, $\vartheta = \pi$ are degenerate cones, namely, plane angular sectors in the plane y = 0. Their semi-angle is arc cos k. Similarly, the surfaces corresponding to $\varphi = 0, \pi, 2\pi$ are plane angular sectors in the plane y = 0 of semi-angle arc cos k'. Consequently, the problem of diffraction by a plane angular sector corresponds to the case $\vartheta_o = \pi$.

The metric coefficients h_1, h_2, h_3 in the expression for the line element

$$ds^2 = h_1^2\, dr^2 + h_2^2\, d\vartheta^2 + h_3^2 d\varphi^2$$

are easily calculated from (2.1). We find

$$\begin{aligned} h_1^2 &= 1 \\ h_2^2 &= r^2 \, \frac{k^2 \sin^2\vartheta + k'^2 \sin^2\varphi}{1 - k^2 \cos^2\vartheta} \qquad (2.2) \\ h_3^2 &= r^2 \, \frac{k^2 \sin^2\vartheta + k'^2 \sin^2\varphi}{1 - k'^2 \cos^2\varphi} \end{aligned}$$

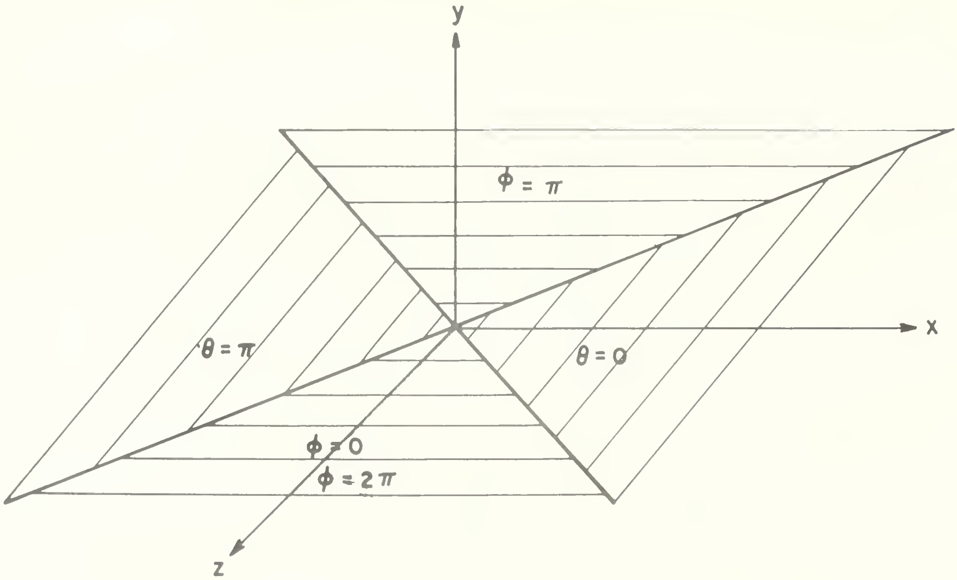

Figure 3

The singular sectors and lines in the plane $y = 0$.

Using (2.2) we obtain the contravariant form of the gradient:

$$\nabla = \vec{i}_r \frac{\partial}{\partial r} + \vec{i}_\vartheta \frac{\sqrt{1 - k^2\cos^2\vartheta}}{r\sqrt{k^2\sin^2\vartheta + k'^2\sin^2\varphi}} \frac{\partial}{\partial\vartheta} + \vec{i}_\varphi \frac{\sqrt{1 - k'^2\cos^2\varphi}}{r\sqrt{k^2\sin^2\vartheta + k'^2\sin^2\varphi}} \frac{\partial}{\partial\varphi} \qquad (2.3)$$

Here $\vec{i}_r$, $\vec{i}_\vartheta$, $\vec{i}_\varphi$, are unit vectors in the directions in which the indicated variables are increasing.

For the Laplacian we obtain

$$\Delta = \frac{1}{r^2} \frac{\partial}{\partial r} \; r^2 \frac{\partial}{\partial r} \; + \frac{1}{r^2} \Delta^* \qquad (2.4)$$

Here

$$\Delta^* = \frac{1}{k^2\sin^2\vartheta + k'^2\sin^2\varphi} \left[\sqrt{1 - k^2\cos^2\vartheta} \frac{\partial}{\partial\vartheta} \left(\sqrt{1 - k^2\cos^2\vartheta} \frac{\partial}{\partial\vartheta} \right) \right. \qquad (2.4')$$

$$\left. + \sqrt{1 - k'^2\cos^2\varphi} \frac{\partial}{\partial\varphi} \left(\sqrt{1 - k'^2\cos^2\varphi} \frac{\partial}{\partial\varphi} \right) \right]$$

Before formulating our diffraction problem in sphero-conal coordinates, we must derive some additional 'matching' conditions, which arise because of the peculiarities of the coordinate system. The first of these is the familiar kind of condition that occurs in spherical coordinates, namely, periodicity with respect to the azimuthal angle. It is evident in this case that any function $u(r,\vartheta,\varphi)$ which is a single-valued function of the point (x,y,z) must be periodic in φ with period 2π, if we extend the range of φ in (2.1) to $-\infty < \varphi < \infty$.

We note next, that the surfaces $\vartheta = 0$ and $\vartheta = \pi$ are singular coordinate surfaces, since every point therein (except on the edges $\varphi = \pi$) is described in two ways. The two descriptions are $(r,0,\varphi)$ or $(r,0,2\pi-\varphi)$ on the sector $\vartheta = o$ and (r,π,φ) or $(r,\pi,2\pi-\varphi)$ on the sector $\vartheta = \pi$. (See Figures 3 and 4.) Therefore we must have for all φ, r

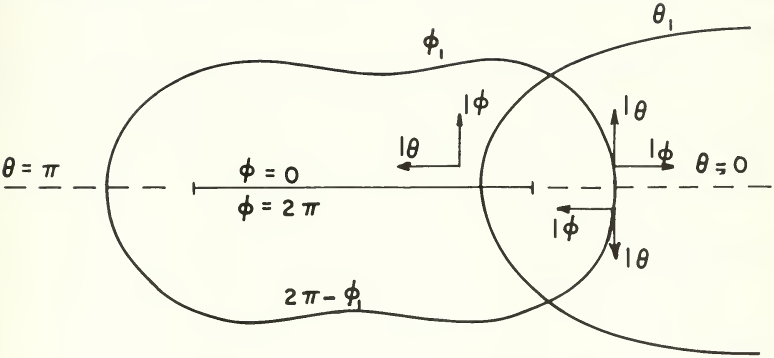

Figure 4

The intersections of the ϑ and φ coordinate surfaces on a sphere around the origin as seen looking toward the origin from the positive z-axis. The arrows indicate the directions in which the variables are increasing.

$$u(r,0,\varphi) = u(r,0,2\pi-\varphi)$$

By taking into account the periodicity with respect to φ, this can be written as

$$u(r,0,\varphi) = u(r,0,-\varphi) \tag{2.5}$$

We have placed our 'obstacle' C so that the sector $\vartheta = \pi$ is either outside the region R, or forms its boundary, if C is a plane angular sector. Therefore u need not be the same on both sides of the sector, so we do not get a condition for $\vartheta = \pi$.

We also require that grad u be the same for both descriptions of the sector $\vartheta = 0$. It is seen in Figure 4, and easily verified analytically* that $\vec{i}_\vartheta$ and $\vec{i}_\varphi$ reverse direction when crossing the sector $\vartheta = 0$. Hence we must have, for grad u to be continuous, (taking into account the periodicity in φ),

$$\begin{aligned} u_\vartheta(r,0,\varphi) &= -u_\vartheta(r,0,-\varphi) \\ u_\varphi(r,0,\varphi) &= -u_\varphi(r,0,-\varphi) \end{aligned} \tag{2.6}$$

The second equation is seen to be a consequence of (2.5) and hence may be omitted.

There are also matching conditions for grad u at the lines on the plane $y = 0$, common to adjacent sectors. From Figure 4 we see that on opposite sides of the line $\vartheta = 0$, $\varphi = 0$, for example, the unit vector $\vec{i}_\vartheta$ has perpendicular directions; the same is true of $\vec{i}_\varphi$. This holds also for the other three such lines. We require that as we approach the line $\vartheta = 0$, $\varphi = 0$, for example, along either of the sectors $\vartheta = 0$ or $\varphi = 0$, we obtain the same limit for grad u. For the components of grad u perpendicular to the plane $y = 0$, we get, approaching the line $\vartheta = 0$, $\varphi = 0$, along the sectors $\varphi = 0$ and $\vartheta = 0$ respectively

---

[*] The unit vectors $\vec{i}_r$, $\vec{i}_\vartheta$, $\vec{i}_\varphi$ are easily computed from (2.1) since $\vec{i}_\vartheta$, for example, has the direction of

$$\frac{\partial\vec{r}}{\partial\vartheta} = \frac{\partial}{\partial\vartheta}\left[\vec{i}x(r,\vartheta,\varphi) + \vec{j}y(r,\vartheta,\varphi) + \vec{k}z(r,\vartheta,\varphi)\right]$$

$$\lim_{\vartheta \to 0+} \frac{1}{h_3(r,\vartheta,0)} u_\varphi(r,\vartheta,0) = \lim_{\varphi \to 0+} \frac{1}{h_2(r,0,\varphi)} u_\vartheta(r,0,\varphi)$$

For the components parallel to $y = 0$ and tangent to the sphere $r = \text{const.}$ through the point we get

$$\lim_{\vartheta \to 0+} \frac{1}{h_2(r,\vartheta,0)} u_\vartheta(r,\vartheta,0) = -\lim_{\varphi \to 0+} \frac{1}{h_3(r,0,\varphi)} u_\varphi(r,0,\varphi)$$

Taking the derivative of (2.5) with respect to φ, and using (2.6) and the condition of periodicity with respect to φ, we get

$$u_\vartheta(r,0,0) = u_\varphi(r,0,0) = 0$$

The first two of the last three conditions can now be rewritten as

$$\lim_{\vartheta \to 0+} \frac{\vartheta}{h_3(r,\vartheta,0)} \frac{u_\varphi(r,\vartheta,0) - u_\varphi(r,0,0)}{\vartheta} = \lim_{\varphi \to 0+} \frac{\varphi}{h_2(r,0,\varphi)} \frac{u_\vartheta(r,0,\varphi)-u_\vartheta(r,0,0)}{\varphi}$$

$$\lim_{\vartheta \to 0+} \frac{\vartheta}{h_2(r,\vartheta,0)} \frac{u_\vartheta(r,\vartheta,0) - u_\varphi(r,0,0)}{\vartheta} = -\lim_{\varphi \to 0+} \frac{\varphi}{h_3(r,0,\varphi)} \frac{u_\varphi(r,0,\varphi)-u_\varphi(r,0,0)}{\varphi}$$

Upon evaluating these limits, the first condition turns out to be $(k,k' \neq 0)$

$$u_{\varphi\vartheta}(r,0,0) = u_{\vartheta\varphi}(r,0,0)$$

This equation is trivially satisfied at any point in a neighborhood of which both u_ϑ, u_φ exist, and at which one of the mixed derivatives exists and is continuous. The second condition, however, is not trivial: we have, for $k,k' \neq 0$,

$$k'^2 u_{\vartheta\vartheta}(r,0,0) + k^2 u_{\varphi\varphi}(r,0,0) = 0 \tag{2.7}$$

In essentially the same way, we get the same condition on the line $\vartheta = 0$, $\varphi = \pi$.

In our problem, the other two singular lines are either outside the region R, i.e. inside the 'obstacle' C, or form its edges. In the latter case, grad u is not necessarily continuous at such singularities of the boundary surface; in fact it is generally infinite at an edge. Thus, in either case, we do not get any matching conditions on the other two lines.

Condition (2.7) is a new condition; it does not follow, in general, from conditions (2.5), (2.6) and the periodicity in φ. For example, the function $\cos(\vartheta+\varphi)$ satisfies these conditions, but does not satisfy (2.7). However, if $u(r,\vartheta,\varphi)$ is a solution of the wave equation (or potential, or heat equation) then (2.7) is automatically satisfied. To see this we write the wave equation, using (2.4) for the Laplacian, and we get

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2(k^2\sin^2\vartheta + k'^2\sin^2\varphi)}\Big[(1-k^2\cos^2\vartheta)u_{\vartheta\vartheta}+(1-k'^2\cos^2\varphi)u_{\varphi\varphi}$$

$$+ k^2\sin\vartheta\cos\vartheta u_\vartheta + k'^2\sin\varphi\cos\varphi u_\varphi\Big] + \kappa^2 u = 0$$

Upon cross-multiplying by $k^2\sin^2\vartheta + k'^2\sin^2\varphi$, and letting $\vartheta \to 0$, $\varphi \to 0$, we see that the expression in the brackets must vanish, and we get (2.7). Therefore we need take into account only the matching conditions (2.5), (2.6), and the periodicity in φ.

We can now formulate our diffraction problem in sphero-conal coordinates. We seek functions $G^{(1)}(r,\vartheta,\varphi;\ r',\vartheta',\varphi')$, $G^{(2)}(r,\vartheta,\varphi;\ r',\vartheta',\varphi')$ defined for $r,r' \geq 0$, $0 \leq \vartheta$, $\vartheta' \leq \vartheta_0$, $-\infty < \varphi$, $\varphi' < \infty$, with $(r,\vartheta,\varphi) \neq (r',\vartheta',\varphi' \pm 2m\pi)$, $m = 0,1,2,\ldots$, satisfying the following conditions: $[(r',\vartheta',\varphi')$ fixed$]$:

a) The wave equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial G^{(1,2)}}{\partial r}\right)+\frac{1}{r^2}\Delta^* G^{(1,2)}+\kappa^2 G^{(1,2)}=-\frac{\delta(r-r')\delta(\vartheta-\vartheta')\delta(\varphi-\varphi')}{r^2\sigma} \qquad (2.8)$$

b) Except where $\vartheta = \vartheta'$, $r = r'$, and $\varphi = \varphi' \pm 2m\pi$, $m = 0,1,2,\ldots$, $G^{(1,2)}$ is of class $C^{(2)}$ in the range $0 \leqq \vartheta < \vartheta_o$, $r \neq 0$, and of class $C^{(o)}$, $0 \leqq \vartheta \leqq \vartheta_o$.

c) The boundary conditions

$$G^{(1)}(r,\vartheta_o,\varphi;r',\vartheta',\varphi') = 0 \qquad (2.9)$$

$$G_\vartheta^{(2)}(r,\vartheta_o,\varphi;r',\vartheta',\varphi') = 0 \qquad (2.10)$$

d) The radiation condition

$$\lim_{r\to\infty} r^2 \int_0^{\vartheta_o}\int_0^{2\pi}\left|\frac{\partial G^{(1,2)}}{\partial r} - i\kappa G^{(1,2)}\right|^2 \sigma d\varphi d\vartheta = 0$$

e) The matching conditions

$$G^{(1,2)}(r,\vartheta,\varphi;r',\vartheta',\varphi') = G^{(1,2)}(r,\vartheta,\varphi+2\pi;r',\vartheta',\varphi')$$

$$G^{(1,2)}(r,0,\varphi;r',\vartheta',\varphi') = G^{(1,2)}(r,0,-\varphi;r',\vartheta',\varphi') \qquad (2.11)$$

$$G_\vartheta^{(1,2)}(r,0,\varphi;r',\vartheta',\varphi') = -G_\varphi^{(1,2)}(r,0,-\varphi;r',\vartheta',\varphi') \qquad (2.12)$$

In the above Δ^* is given in (2.4') and σ is defined by

$$\sigma = \frac{k^2\sin^2\vartheta + k'^2\sin^2\varphi}{\sqrt{1-k^2\cos^2\vartheta}\ \sqrt{1-k'^2\cos^2\varphi}} = \frac{1}{r^2}\ h_1h_2h_3$$

Then $r^2\sigma dr d\vartheta d\varphi$ is the element of volume.

3. Green's functions.

We consider the homogeneous equation corresponding to (2.8), namely

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\Delta^* u + \kappa^2 u = 0 \qquad (3.1)$$

Setting $u = R(r)V(\vartheta,\varphi)$ in (3.1) and multiplying by $\frac{r^2}{u}$, we get

$$\frac{1}{R}\frac{d}{dr}(r^2R') + \frac{\Delta^* V}{V} + \kappa^2 r^2 = 0$$

As in the case of spherical coordinates, we write the separation constant as $\nu(\nu+1)$:

$$\frac{1}{R}\frac{d}{dr}(r^2R') + \kappa^2 r^2 = -\frac{\Delta^* V}{V} = \nu(\nu+1)$$

Then

$$\frac{d}{dr}(r^2R') + \left[\kappa^2 r^2 - \nu(\nu+1)\right] R = 0 \tag{3.2}$$

$$\Delta^* V + \nu(\nu+1)V = 0 \tag{3.3}$$

The solutions of (3.2) are the spherical Bessel functions. In particular we shall employ the solutions

$$\begin{aligned} j_\nu(\kappa r) &= \sqrt{\frac{\pi}{2\kappa r}}\; J_{\nu+1/2}(\kappa r) \\ h_\nu(\kappa r) &= \sqrt{\frac{\pi}{2\kappa r}}\; H^{(1)}_{\nu+1/2}(\kappa r) \end{aligned} \tag{3.4}$$

The first function satisfies the condition of continuity at the vertex; the second satisfies the radiation condition. The values of ν are determined by the eigenvalues $\nu(\nu+1)$ of (3.3) with one of the boundary conditions $V(\vartheta_o,\varphi) = 0$, or $V_\vartheta(\vartheta_o,\varphi) = 0$.

Next, equation (3.3) separates as follows. Let $V(\vartheta,\varphi) = \Theta(\vartheta)\,\Phi(\varphi)$. Substituting in (3.3) we get the Sturm-Liouville equations

$$(1-k^2\cos^2\vartheta)^{1/2}\frac{d}{d\vartheta}\left[(1-k^2\cos^2\vartheta)^{1/2}\Theta'(\vartheta)\right] + \left[\nu(\nu+1)k^2\sin^2\vartheta+\mu\right]\Theta(\vartheta) = 0 \tag{3.5}$$

$$(1-k'^2\cos^2\varphi)^{1/2}\frac{d}{d\varphi}\left[1-k'^2\cos^2\varphi)^{1/2}\,\Phi'(\varphi)\right] + \left[\nu(\nu+1)k'^2\sin^2\varphi-\mu\right]\Phi(\varphi) = 0 \tag{3.6}$$

Here μ is the separation constant. We have in each case the trigonometric form of the Lamé differential equation. We now apply our matching conditions to the eigenfunction $V(\vartheta,\varphi) = \Theta(\vartheta)\,\Phi(\varphi)$. The periodicity condition implies that $\Phi(\varphi)$ must have period 2π. Conditions (2.11) and (2.12) imply that

$$\Theta(0)\,\Phi(\varphi) = \Theta(0)\,\Phi(-\varphi) \tag{3.7}$$

$$\Theta'(0)\,\Phi(\varphi) = -\,\Theta'(0)\,\Phi(-\varphi) \tag{3.8}$$

Equation (3.7) states that either $\Theta(0) = 0$ or $\Phi(\varphi) = \Phi(-\varphi)$, while (3.8) states that either $\Theta'(0) = 0$ or $\Phi(\varphi) = -\Phi(-\varphi)$. We cannot have both $\Theta(0) = 0$ and $\Theta'(0) = 0$, since this would mean that $\Theta(\vartheta) \equiv 0$. Moreover, since $\Phi(\varphi)$ cannot be both even and odd, i.e. identically zero, we must have either

$$\Theta(0) = 0 \qquad \text{and} \qquad \Phi(\varphi) = -\,\Phi(-\varphi)$$

or

$$\Theta'(0) = 0 \qquad \text{and} \qquad \Phi(\varphi) = \Phi(-\varphi)$$

Since (3.5) is invariant under a change of sign of the independent variable, we see that if $\Theta'(0) = 0$, $\Theta(\vartheta)$ is an even solution of (3.5), and if $\Theta(0) = 0$, then $\Theta(\vartheta)$ is an odd solution. Thus, the matching conditions imply that the factors $\Theta(\vartheta)$, $\Phi(\varphi)$ of the eigenfunction $V(\vartheta,\varphi)$ are either both odd or both even.

We now define solutions $ye(\vartheta;k,\nu,\mu)$, $yo(\vartheta;k,\nu,\mu)$ of equation (3.5) satisfying the initial conditions

$$\begin{aligned} ye(0;k,\nu,\mu) &= 1 \qquad & yo(0;k,\nu,\mu) &= 0 \\ ye'(0;k,\nu,\mu) &= 0 \qquad & yo'(0;k,\nu,\mu) &= 1 \end{aligned} \tag{3.9}$$

We shall call $ye(\vartheta;k,\nu,\mu)$, $yo(\vartheta;k,\nu,\mu)$ the even and odd Lamé functions, respectively. In this notation, the associated solutions of (3.6) are, then, $ye(\varphi;k',\nu,-\mu)$ and $yo(\varphi;k',\nu,-\mu)$.

It is a consequence of the Sturm-Liouville theory that for any ν there is a discrete set of values $\mu_1, \mu_2, \ldots$ such that the solutions $ye(\varphi;k',\nu,-\mu_1)$ of (3.6) are periodic with period 2π (half of these will actually have period π); similarly, there is a set $\tilde{\mu}_1, \tilde{\mu}_2, \ldots$ corresponding to odd periodic solutions with period 2π. Together, these sets of functions comprise a complete orthogonal set over the interval $[-\pi,\pi]$, and any twice continuously differentiable function can be expanded in a uniformly convergent Fourier series' of these functions (see [9] for example). These functions are the periodic Lamé functions and have been investigated in several papers (Ince[6,7]; see also Erdelyi[8]*).

On the other hand, the functionsof ϑ satisfying one or the other of the boundary conditions $ye(\vartheta_o;k,\nu,\mu) = 0$ or $ye'(\vartheta_o,k,\nu,\mu) = 0$, are in general not periodic. (The case $\vartheta_o = \pi$, i.e. when C is a plane angular sector, is an exception.**) Again, by the Sturm-Liouville theory, for any ν, there is a sequence of μ's associated with even solutions forming a complete orthogonal set, for which the above expansion theorem holds. A similar statement holds with respect to odd solutions.

However, we have here the two parameter eigenvalue problem of determining all the pairs $(\nu_{en}^{(1,2)},\mu_{en}^{(1,2)})$ for which the solutions $ye(\varphi,k',\nu_{en}^{(1,2)},-\mu_{en}^{(1,2)})$ are 2π periodic and, simultaneously, the solutions $ye(\vartheta;k,\nu_{en}^{(1,2)},\mu_{en}^{(1,2)})$ satisfy

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[*] It is interesting to note that Ince obtained the trigonometric form of the Lamé equation from the standard elliptic form by a transformation of the independent variable, for the purpose of simplifying the analysis; on the other hand, we have obtained the trigonometric form directly, due to the form of the coordinate system which we have adopted.

[**] For this case the functions $ye(\vartheta;k,\nu,\mu)$, $yo(\vartheta;k,\nu,\mu)$ are periodic with period π, 2π or 4π. A detailed discussion is given in [1].

the boundary conditions

$$\begin{aligned} &ye(\vartheta_o;k,\nu_{en}^{(1)},\mu_{en}^{(1)}) = 0 \\ &ye'(\vartheta_o;k,\nu_{en}^{(2)},\mu_{en}^{(2)}) = 0 \end{aligned} \tag{3.10}$$

Similarly we seek the pairs $(\nu_{on}^{(1,2)},\mu_{on}^{(1,2)})$ for which the odd solutions satisfy the same conditions. We show in Appendix II that the Lamé products

$$\begin{aligned} &ye(\vartheta;k,\nu_{en}^{(1,2)},\mu_{en}^{(1,2)})\, ye(\varphi,k',\nu_{en}^{(1,2)},-\mu_{en}^{(1,2)}) \ , \\ &yo(\vartheta;k,\nu_{on}^{(1,2)},\mu_{on}^{(1,2)})\, yo(\varphi;k',\nu_{on}^{(1,2)},-\mu_{on}^{(1,2)}) \end{aligned} \tag{3.11}$$

constitute a complete orthogonal set of eigenfunctions of (3.3) for each boundary condition.

We now proceed to the construction of the required Green's functions.

Set

$$G^{(1,2)} = \sum_{n=1}^{\infty} R_{en}^{(1,2)}(r) ye(\vartheta;k,\nu_{en}^{(1,2)},\mu_{en}^{(1,2)}) ye(\varphi;k',\nu_{en}^{(1,2)},-\mu_{en}^{(1,2)}) \tag{3.12}$$

$$+\sum_{n=1}^{\infty} R_{on}^{(1,2)}(r)\ yo(\vartheta;k,\nu_{on}^{(1,2)},\mu_{on}^{(1,2)}) yo(\varphi;k',\nu_{on}^{(1,2)},-\mu_{on}^{(1,2)})$$

We substitute (3.12) into (2.8) and use (3.3) to simplify the resulting equation. Then we multiply by $\sigma ye(\vartheta;k,\nu_{em}^{(1,2)},\mu_{em}^{(1,2)}) ye(\varphi;k',\nu_{em}^{(1,2)},-\mu_{em}^{(1,2)})$, and integrate over the region $-\pi \leq \varphi \leq \pi$, $0 \leq \vartheta \leq \vartheta_o$. Using the orthogonality of the Lamé products with respect to the weight function σ, we get

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR_{em}^{(1,2)}}{dr}\right) + \left[\kappa^2 - \frac{\nu_{em}^{(1,2)}(\nu_{em}^{(1,2)}+1)}{r^2}\right] R_{em}^{(1,2)} \tag{3.13}$$

$$= - \frac{ye(\vartheta';k,\nu_{em}^{(1,2)},\mu_{em}^{(1,2)})\ ye(\varphi';k',\nu_{em}^{(1,2)},-\mu_{em}^{(1,2)})\delta(r-r')}{r^2 N_{em}^{(1,2)}}$$

Here

$$N_{em}^{(1,2)} = \int_{-\pi}^{\pi}\int_{0}^{\vartheta_0}\left[ye(\vartheta;k,\nu_{em}^{(1,2)},\mu_{em}^{(1,2)})ye(\varphi;k',\nu_{em}^{(1,2)},-\mu_{em}^{(1,2)})\right]^2 \sigma d\vartheta d\varphi \tag{3.14}$$

The function $R_{em}^{(1,2)}$ satisfies the spherical Bessel equation

$$\frac{d}{dr}\left(r^2\ \frac{dR_{em}^{(1,2)}}{dr}\right) + \left[\kappa^2 r^2 - \nu_{em}^{(1,2)}(\nu_{em}^{(1,2)} + 1)\right]R_{em}^{(1,2)} = 0 \tag{3.15}$$

except at $r = r'$. There it is continuous, with a jump in the derivative given by

$$r^2\ \frac{dR_{em}^{(1,2)}}{dr}\Bigg|_{r'-}^{r'+} = -\ \frac{ye(\vartheta';k,\nu_{em}^{(1,2)},\mu_{em}^{(1,2)})ye(\varphi';k',\nu_{em}^{(1,2)},-\mu_{em}^{(1,2)})}{N_{em}^{(1,2)}} \tag{3.16}$$

Since $R_{em}^{(1,2)}(r)$ is to be continuous at the origin, it must be proportional there to $j_{\nu_{em}^{(1,2)}}(\kappa r)$. To satisfy the radiation condition it must be proportional to $h_{\nu_{em}^{(1,2)}}(\kappa r)$ for large r. From the continuity at r', we get

$$R_{em}^{(1,2)} = B_{em}^{(1,2)} j_{\nu_{em}^{(1,2)}}(\kappa r_<)h_{\nu_{em}^{(1,2)}}(\kappa r_>) \tag{3.17}$$

Here we have used the common notation

$$r_< = \min(r,r')$$

$$r_> = \max(r,r')$$

When we apply the jump condition, (3.16), to the left side of (3.17), we get

$$B_{em}^{(1,2)} = -\ \frac{ye(\vartheta';k,\nu_{em}^{(1,2)},\mu_{em}^{(1,2)})ye(\varphi';k',\nu_{em}^{(1,2)},-\mu_{em}^{(1,2)})}{\kappa N_{em}^{(1,2)} r'^2\left[W(j_{\nu_{em}^{(1,2)}},\ h_{\nu_{em}^{(1,2)}})\right]_{\kappa r'}} \tag{3.18}$$

Here $W(j,h)$ is the Wronskian. We have

$$W(j_{\nu_{em}^{(1,2)}},\ h_{\nu_{em}^{(1,2)}})\Big|_{\kappa r} = \frac{i}{(\kappa r')^2} \tag{3.19}$$

We proceed in exactly the same way to find the coefficients $R^{(1,2)}_{om}(r)$ of the odd Lamé products.

Substituting for $R^{(1,2)}_{em}(r)$ and $R^{(1,2)}_{om}(r)$ in (3.12), we get for the Green's functions $(p=1,2)$

$$G^{(p)}(r,\vartheta,\varphi;r',\vartheta',\varphi') = i\kappa \sum_{n=1}^{\infty} \frac{1}{N^{(p)}_{en}}\, j_{\nu^{(p)}_{en}}(\kappa r_<)\, h_{\nu^{(p)}_{en}}(\kappa r_>) \tag{3.20}$$

$$ye(\vartheta;k,\nu^{(p)}_{en},\mu^{(p)}_{en})\,ye(\vartheta';k,\nu^{(p)}_{en},\mu^{(p)}_{en})\; ye(\varphi;k',\nu^{(p)}_{en},-\mu^{(p)}_{en})\,ye(\varphi';k',\nu^{(p)}_{en},-\mu^{(p)}_{en})$$

$$+\, i\kappa \sum_{n=1}^{\infty} \frac{1}{N^{(p)}_{on}}\, j_{\nu^{(p)}_{on}}(\kappa r_<)\, h_{\nu^{(p)}_{on}}(\kappa r_>)$$

$$yo(\vartheta;k,\nu^{(p)}_{on},\mu^{(p)}_{on})\,yo(\vartheta';k,\nu^{(p)}_{on},\mu^{(p)}_{on})\; yo(\varphi;k',\nu^{(p)}_{on},-\mu^{(p)}_{on})\,yo(\varphi';k',\nu^{(p)}_{on},-\mu^{(p)}_{on})$$

Here, $N^{(1,2)}_{en}$ is defined by (3.14), and $N^{(1,2)}_{on}$ is defined similarly by

$$N^{(1,2)}_{on} = \int_{-\pi}^{\pi}\int_{0}^{\vartheta_0} \left[yo(\vartheta;k,\nu^{(1,2)}_{on},\mu^{(1,2)}_{on})\,yo(\varphi;k',\nu^{(1,2)}_{on},-\mu^{(1,2)}_{on})\right]^2 \sigma\, d\vartheta\, d\varphi \tag{3.21}$$

The expression (3.20) for the Green's functions is our final result.

Appendix I. The sphero-conal coordinate system.

By analogy to confocal (ellipsoidal) coordinates, we consider the set of 'confocal cones'

$$\frac{x^2}{\xi} + \frac{y^2}{\xi - b^2} + \frac{z^2}{\xi - c^2} = 0 \qquad 0 < b^2 < c^2 \tag{I-1}$$

These surfaces are two-napped elliptical cones with a common vertex at the origin. Exactly as in the case of confocal coordinates, we can easily prove that if two distinct surfaces of (I-1) have in common a point other than the origin, they are orthogonal at that point. To do so we define F by

$$F(x,y,z;\xi) = \frac{x^2}{\xi} + \frac{y^2}{\xi - b^2} + \frac{z^2}{\xi - c^2}$$

Let (x_1,y_1,z_1) be a point common to the surfaces

$$F_1(x,y,z) = F(x,y,z;\xi_1) = 0$$

$$F_2(x,y,z) = F(x,y,z;\xi_2) = 0 \qquad \xi_1 \neq \xi_2$$

Then we have, at (x_1,y_1,z_1),

$$\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial x} + \frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial z} = \frac{4}{\xi_2 - \xi_1}\left[F_1(x_1,y_1,z_1) - F_2(x_1,y_1,z_1)\right] = 0.$$

This proves the orthogonality.

If x,y,z are fixed, (I-1) becomes a quadratic equation in ξ and consequently has at most two distinct roots. Thinking of $F(x,y,z;\xi)$ as a function of ξ, we easily see that it has at least one zero ν^2 between 0 and b^2, and a zero μ^2 between b^2 and c^2. Therefore it has exactly one such zero in each of these intervals. Hence through any point $(x,y,z) \neq (0,0,0)$ there are exactly two cones

$$\frac{x^2}{\mu^2} + \frac{y^2}{\mu^2-b^2} - \frac{z^2}{c^2-\mu^2} = 0$$

$$\frac{x^2}{\nu^2} - \frac{y^2}{b^2-\nu^2} - \frac{z^2}{c^2-\mu^2} = 0$$

We shall use these two sets of cones as coordinate surfaces. As our third set of coordinate surfaces, orthogonal to the two sets of cones, we take the system of concentric spheres centered at the origin. Thus we consider the transformation of coordinates from (x,y,z) to (r,μ,ν)

$$\begin{aligned} &\text{a)} \quad r^2 = x^2 + y^2 + z^2 && c > b > 0 \\ &\text{b)} \quad \frac{x^2}{\mu^2} + \frac{y^2}{\mu^2-b^2} - \frac{z^2}{c^2-\mu^2} = 0 && 0 \leq \nu^2 \leq b^2 \leq \mu^2 \leq c^2 \\ &\text{c)} \quad \frac{x^2}{\nu^2} - \frac{y^2}{b^2-\nu^2} - \frac{z^2}{c^2-\nu^2} = 0 && \end{aligned} \tag{I-2}$$

These equations give μ^2, ν^2, r^2 in terms of x,y,z, as solution of quadratic equations.

To get the inverse transformation, we use the fact that μ^2 and ν^2 are the roots of the equation (in ξ)

$$\frac{x^2}{\xi} + \frac{y^2}{\xi-b^2} + \frac{z^2}{\xi-c^2} = \frac{p(\xi)}{\xi(\xi-b^2)(\xi-c^2)} = 0$$

We note that $p(\xi)$ is a quadratic expression in ξ with leading coefficient $x^2 + y^2 + z^2 = r^2$, and roots μ^2, ν^2. Hence

$$\frac{x^2}{\xi} + \frac{y^2}{\xi-b^2} + \frac{z^2}{\xi-c^2} = \frac{r^2(\xi-\nu^2)(\xi-\mu^2)}{\xi(\xi-b^2)(\xi-c^2)} \tag{I-3}$$

Upon multiplying (I-3) by $\xi, \xi-b^2, \xi-c^2$, and setting $\xi=0$, $\xi=b^2$, $\xi=c^2$, respectively, in the resulting identities, we get the inverse transformation

$$\text{a)}\quad x^2 = \frac{r^2\nu^2\mu^2}{b^2c^2}$$

$$\text{b)}\quad y^2 = \frac{r^2(\mu^2-b^2)(b^2-\nu^2)}{b^2(c^2-b^2)} \qquad \begin{array}{l} 0 \leqq \nu^2 \leqq b^2 \\ b^2 \leqq \mu^2 \leqq c^2 \end{array} \tag{I-4}$$

$$\text{c)}\quad z^2 = \frac{r^2(c^2-\mu^2)(c^2-\nu^2)}{c^2(c^2-b^2)}$$

For a given set of values r,μ,ν the surfaces (I-2), i.e. the sphere a) and the two-napped cones b) and c), intersect in eight points, in general. We attempt to achieve a 1 - 1 correspondence between the points (x,y,z) and the coordinates (r,μ,ν) by proceeding as follows. First we restrict (r,μ,ν) to the ranges $r \geqq 0$, $b \leqq \mu \leqq c$ and $-b \leqq \nu \leqq b$. Then r,μ,ν are uniquely determined by x,y,z. To determine x,y,z from r,μ,ν we must specify which square roots are to be taken in solving (I-4). In (I-4a) we take that square root which has the same sign as ν.

We cannot remove the remaining ambiguities in the signs of y and z in the same way. Therefore we introduce instead of μ the new variable $\lambda = \sqrt{c^2-\mu^2}$. We assign to λ the range $-\sqrt{c^2-b^2} \leqq \lambda \leqq \sqrt{c^2-b^2}$. Now we define z by taking that square root in (I-4c) which has the same sign as λ.

In order to specify which root is to be taken in (I-4b) it is convenient to replace ν and λ by the two new variables ϑ and φ, defined by

$$\nu = b\cos\vartheta \qquad 0 \leqq \vartheta \leqq \pi$$

$$\lambda = \sqrt{c^2-\mu^2} = \sqrt{c^2-b^2}\ \cos\varphi \qquad 0 \leqq \varphi \leqq 2\pi$$

Then

$$\sqrt{\mu^2-b^2} = \sqrt{c^2-b^2}\ \sin\varphi$$

Now, upon setting $k = \frac{b}{c}$, we get from (I-4) the formulas (2.1). These formulas give the desired 1 - 1 correspondence between the coordinates (x,y,z) and (r,ϑ,φ)

except for the singular coordinate surfaces $\vartheta = 0$, $\vartheta = \pi$ (which are discussed in section 2).

Upon substituting for ν and $\frac{b}{c}$ in (I-2c) we see that the surface $\vartheta = \vartheta_o$ is a single nappe of the cone

$$\frac{y^2}{\sin^2\vartheta_o} + \frac{z^2}{(\frac{1}{k^2} - 1)+\sin^2\vartheta_o} = \frac{x^2}{\cos^2\vartheta_o} \tag{I-5}$$

The nappe of the cone encloses the **positive or negative x-axis depending** on whether $0 < \vartheta_o < \frac{\pi}{2}$ or $\frac{\pi}{2} < \vartheta_o < \pi$. The cross section of the cone (I-5) in the plane $x = x_1$ is given by the ellipse

$$\frac{y^2}{x_1^2\tan^2\vartheta_o} + \frac{z^2}{x_1^2\tan^2\vartheta_o + \frac{x_1^2}{\cos^2\vartheta_o}\left(\frac{1}{k^2} - 1\right)} = 1 \tag{I-6}$$

If (I-5) is the boundary cone C, and $x = x_1$ the plane of its directrix, the latter is given by (I-6). If ϵ is the semi-focal angle of C, then $x_1\tan\epsilon$ is the semi-focal distance of the directrix, and from (I-6) we get

$$x_1\tan\epsilon = \frac{x_1}{\cos\vartheta_o}\sqrt{\frac{1}{k^2} - 1}$$

This gives us k in terms of the boundary cone C.

Appendix II. Orthogonality and completeness of the Lamé products.

Let $V_1(\vartheta,\varphi)$, $V_2(\vartheta,\varphi)$ be any two of the Lamé products of (3.11), satisfying the same boundary condition. We shall first prove that, unless $V_1 \equiv V_2$, they are orthogonal, i.e.

$$\int_{-\pi}^{\pi}\int_{0}^{\pi} V_1(\vartheta,\varphi)V_2(\vartheta,\varphi)\sigma d\vartheta d\varphi = 0 \qquad \text{(II-1)}$$

First, we note that if one of the products V_1, V_2 is odd and the other even, (II-1) follows immediately; for then $V_1V_2\sigma$ would be an odd function of φ, and, integrating first with respect to φ, we have the result. We next assume that

$$V_1 = ye(\vartheta;k,\nu^{(1)}_{em_1},\mu^{(1)}_{em_1})ye(\varphi;k',\nu^{(1)}_{em_1},-\mu^{(1)}_{em_1})$$

$$V_2 = ye(\vartheta;k,\nu^{(1)}_{em_2},\mu^{(1)}_{em_2})ye(\varphi;k',\nu^{(1)}_{em_2},-\mu^{(1)}_{em_2})$$

(The proof proceeds in exactly the same way if V_1,V_2 are odd Lamé products, or if they satisfy the second boundary condition.) Let $\lambda_i = \nu^{(1)}_{em_i}(\nu^{(1)}_{em_i} + 1)$, $\Theta_i = ye(\vartheta;k,\nu^{(1)}_{em_i},\mu^{(1)}_{em_i})$ and $\Phi_i = ye(\varphi;k',\nu^{(1)}_{em_i},-\mu^{(1)}_{em_i})$, $i = 1,2$.

From (3.5) we get (II-2)

$$\frac{d}{d\vartheta}\left\{(1 - k^2\cos^2\vartheta)^{1/2}\left[\Theta_1\Theta_2' - \Theta_2\Theta_1'\right]\right\} = \left[(\lambda_1-\lambda_2)k^2\sin^2\vartheta + \mu^{(1)}_{em_1} - \mu^{(1)}_{em_2}\right]\frac{\Theta_1\Theta_2}{\sqrt{1-k^2\cos^2\vartheta}}$$

Upon integrating (II-2) from 0 to ϑ_o, we find that the left side vanishes, since the quantity inside the square bracket vanishes at both end points. This is so at $\vartheta = 0$ because the functions are even and at $\vartheta = \vartheta_o$ because we are assuming the first boundary condition. It is also clear that the square bracket vanishes at the end points for the various other possibilities, i.e. for a rigid boundary or odd functions or both. Consequently we get

$$(\lambda_1-\lambda_2)k^2\int_{o}^{\vartheta_o}\frac{\Theta_1\Theta_2\sin^2\vartheta}{\sqrt{1-k^2\cos^2\vartheta}}d\vartheta = (\mu^{(1)}_{em_2} - \mu^{(1)}_{em_1})\int_{o}^{\vartheta_o}\frac{\Theta_1\Theta_2 d\vartheta}{\sqrt{1-k^2\cos^2\vartheta}} \qquad \text{(II-3)}$$

Similarly, upon using (3.6) we get

$$(\lambda_1-\lambda_2)k'^2 \int_{-\pi}^{\pi} \frac{\Phi_1 \Phi_2 \sin^2\varphi}{\sqrt{1-k'^2\cos^2\varphi}}\, d\varphi = (\mu^{(1)}_{em_1} - \mu^{(1)}_{em_2}) \int_{-\pi}^{\pi} \frac{\Phi_1 \Phi_2}{\sqrt{1-k'^2\cos^2\varphi}}\, d\varphi \tag{II-4}$$

Now, multiplying opposite sides of (II-3) and (II-4), and combining terms we get

$$(\mu^{(1)}_{em_1} - \mu^{(1)}_{em_2})(\lambda_1-\lambda_2) \int_{-\pi}^{\pi}\int_{0}^{\vartheta_0} \Theta_1\Theta_2 \Phi_1 \Phi_2 \,\sigma d\vartheta d\varphi = 0 \tag{II-5}$$

If neither $\mu^{(1)}_{em_1} = \mu^{(1)}_{em_2}$, nor $\lambda_1 = \lambda_2$, then the integral vanishes and we get (II-1). If $\mu^{(1)}_{em_1} = \mu^{(1)}_{em_2}$ but $\lambda_1 \neq \lambda_2$, then the integrals on the left sides of (II-3), (II-4) vanish, and hence the double integral in (II-5) vanishes. Similarly, if $\lambda_1 = \lambda_2$, but $\mu^{(1)}_{em_1} \neq \mu^{(1)}_{em_2}$, then the integrals on the right sides of (II-3), (II-4) vanish, and again the double integral in (II-5) vanishes. This completes the proof.

We show next that the Lamé products (3.11) form a complete set of eigenfunctions for equation (3.3). First, we recall that there exist complete sets of eigenfunctions $\left\{V^{(1)}_n\right\}$, $\left\{V^{(2)}_n\right\}$ with associated eigenvalues $\left\{\nu^{(1)}_n\right\}\left\{\nu^{(2)}_n\right\}$, $n = 1,2,\ldots$ for the first and second boundary value problems respectively. This is proved in [5]. We shall show that every eigenfunction $V^{(1,2)}_m$ can be expressed as a finite sum of Lamé products, and hence that the latter constitute a complete set.

Again, for definiteness, let us consider a particular boundary value problem say the problem of the rigid boundary, i.e. $\partial u/\partial n = 0$ on C. We shall omit the superscripts indicating this. First, we note that if $V_m(\vartheta,\varphi)$ satisfies (3.3), then so does $V_m(\vartheta,-\varphi)$. But

$$V_m(\vartheta,\varphi) = \frac{1}{2}\left[V_m(\vartheta,\varphi) + V_m(\vartheta,-\varphi)\right] + \frac{1}{2}\left[V_m(\vartheta,\varphi) - V_m(\vartheta,-\varphi)\right]$$

Since the first bracketed term is a solution of (3.3) which is even in φ, and the second a solution odd in φ, we can assume without loss of generality that V_m is either even or odd in φ.

Let us assume that $V_m(\vartheta,\varphi)$ is even φ. Let ν_m be its eigenvalue. Then by the remarks following (3.9), V_m can be expanded in an infinite series of periodic Lamé functions $ye(\varphi;k,\nu_m,-\mu_j)$, $j = 1,2,\ldots$. Because of the evenness of V_m, the Fourier coefficients of the odd periodic functions vanish. Let us normalize the functions $ye(\varphi;k,\nu_m,-\mu_j)$, denoting the corresponding normalized function by $\Phi_j(\varphi)$. Thus, we have

$$\int_{-\pi}^{\pi} \frac{\Phi_i\,\Phi_j}{\sqrt{1-k'^2\cos^2\varphi}}\,d\varphi = \delta_{ij} \tag{II-6}$$

Consequently we have

$$V_m(\vartheta,\varphi) = \sum_{j=1}^{\infty} A_j(\vartheta)\,\Phi_j(\varphi) \tag{II-7}$$

Here

$$A_j(\vartheta) = \int_{-\pi}^{\pi} \frac{V_m(\vartheta,\varphi)\,\Phi_j(\varphi)}{\sqrt{1-k'^2\cos^2\varphi}}\,d\varphi \tag{II-8}$$

From (II-8), we get

$$\sqrt{1-k^2\cos^2\vartheta}\,\frac{d}{d\vartheta}\left(\sqrt{1-k^2\cos^2\vartheta}\,\frac{dA_j(\vartheta)}{d\vartheta}\right) \tag{II-9}$$

$$= \int_{-\pi}^{\pi} \sqrt{1-k^2\cos^2\vartheta}\,\frac{\partial}{\partial\vartheta}\left(\sqrt{1-k^2\cos^2\vartheta}\,\frac{\partial V_m(\vartheta,\varphi)}{\partial\vartheta}\right)\frac{\Phi_j(\varphi)}{\sqrt{1-k'^2\cos^2\varphi}}\,d\varphi$$

$$= \int_{-\pi}^{\pi} \left\{-\sqrt{1-k'^2\cos^2\varphi}\,\frac{\partial}{\partial\varphi}\left(\sqrt{1-k'^2\cos^2\varphi}\,\frac{\partial}{\partial\varphi}V_m(\vartheta,\varphi)\right)\right.$$

$$\left. -\,\nu_m(\nu_m+1)(k^2\sin^2\vartheta + k'^2\sin^2\varphi)V_m(\vartheta,\varphi)\right\}\frac{\Phi_j(\varphi)}{\sqrt{1-k'^2\cos^2\varphi}}\,d\varphi$$

In the last equation, (3.3) was used to transform the integrand. Next, we integrate the first term of the integrand twice by parts, and make use of the 2π periodicity of Φ_j and V_m. The entire integral on the right hand side of (II-9) becomes

$$\int_{-\pi}^{\pi} V_m(\vartheta,\varphi) \left\{ - \sqrt{1-k'^2\cos^2\varphi}\, \frac{d}{d\varphi}\left(\sqrt{1-k'^2\cos^2\varphi}\; \frac{d\Phi_j}{d\varphi}\right) - \nu_m(\nu_m+1)\,(k^2\sin^2\vartheta + k'^2\sin^2\varphi)\,\Phi_j \right\} \frac{d\varphi}{\sqrt{1-k'^2\cos^2\varphi}}$$

Using (3-6), this becomes

$$\int_{-\pi}^{\pi} V_m(\vartheta,\varphi) \left[-\mu_j-\nu_m(\nu_m+1)k^2\sin^2\vartheta\right] \frac{\Phi_j(\varphi)}{\sqrt{1-k'^2\cos^2\varphi}}\, d\varphi$$

By (II-8), this is equal to

$$\left[-\mu_j-\nu_m(\nu_m+1)k^2\sin^2\vartheta\right]A_j(\vartheta)$$

Thus, we have shown that $A_j(\vartheta)$ satisfies (3.5):

$$\sqrt{1-k^2\cos^2\vartheta}\;\frac{d}{d\vartheta}\left(\sqrt{1-k^2\cos^2\vartheta}\;\frac{dA_j(\vartheta)}{d\vartheta}\right) + \left[\nu_m(\nu_m+1)k^2\sin^2\vartheta+\mu_j\right]A_j(\vartheta) = 0 \qquad \text{(II-10)}$$

By the matching condition $V_{m\vartheta}(0,\varphi) = -V_{m\vartheta}(0,-\varphi)$, and the evenness of $\Phi_j(\varphi)$ in φ, (II-8) gives us

$$A_j'(0) = 0 \qquad \text{(II-11)}$$

By the rigid boundary condition $V_{m\vartheta}(\vartheta_o,\varphi) = 0$, (II-8) gives us

$$A_j'(\vartheta_o) = 0 \qquad \text{(II-12)}$$

Hence, from (II-10), (II-11), (II-12)

$$A_j(\vartheta) = c_j\, ye(\vartheta;k,\nu_m,\mu_j)$$

Thus every term $A_j(\vartheta)\ \Phi_j(\varphi)$ of the sum in (II-7) is an even Lamé product in (3.11).

Finally, there can be only a finite number of terms in the sum (II-7). First, we note that there can be only a finite number, say q, of independent terms, since each product in the sum is an eigenfunction belonging to the eigenvalue ν_m, and an eigenvalue of (3.3) can have only a finite multiplicity.* Second, there can be no more than q terms altogether, since any q+1 terms satisfy a linear relation

$$\sum_{j=1}^{q+1} C_j A_j(\vartheta)\ \Phi_j\ (\varphi) = 0 \qquad \text{(II-13)}$$

But we can find some ϑ for which all the $A_j(\vartheta)$ are different from zero; for this value of ϑ, (II-13) constitutes a linear relationship between the $\Phi_j(\varphi)$, which is impossible.

If $V_m(\vartheta,\varphi)$ is an odd function of φ, the proof proceeds in the same way except that the odd Lamé functions are used, and (II-11) is replaced by $A_j(0) = 0$ which follows from the other matching condition $V_m(0,\varphi) = V_m(0,-\varphi)$.

The above proofs of the orthogonality and completeness of the Lamé products are based on related proofs presented in [11] and [10] repectively, for the eigenfunctions of a complete ellipsoid.

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